1. Consider the velocity field:

$$u = \frac{A}{x}, v = \frac{Ay}{x^2}, A = 2.$$

Obtain the streamline in the form $y(x)$ which passes through the point $(1, 3)$.

Calculate the time required to move from x=1 to x=2 in the flow field.

Obtain the expression for the streakline of the flow from point (1,3).

Solution: Recall that since the stream follows the slopes of the velocity, we should use the equation

$$\frac{dy}{dx} = \frac{v}{u}$$

By plugging in the expressions for u and v into this, the equation becomes

$$\frac{dy}{dx} = \frac{y}{x}$$

. Separating variables and integrating gives

$$\ln \frac{y}{y_0} = \ln \frac{x}{x_0}$$

which makes the final expression after plugging in $y = 3x$.

For the second part, we need the pathline to find the expression for x as a function of time.

$$\frac{dx}{dt} = \frac{A}{x}$$

Integrating this between x_0 and x, and t_0 and t, gives the expression

$$0.5(x^2 - x_0^2) = A(t - t_0) = A(\Delta t)$$

Plugging in $x_0 = 1$ and $x = 3$, we get $\Delta t = 0.75$.

Lastly for the streakline, first let us write the general path line expressions. From the previous expression, the integral to find the pathline, we use the dx/dt expression and integration it without x_0 and t_0 in mind. This gives

$$\frac{1}{2}x^2 = At + C, x(t) = \sqrt{2At + C}.$$

Since all points must start at (1,3), we will say that the intial condition of the pathline is also (1,3) at t=0. Plugging in the initial condition (1,3) at t_0, the expression becomes

$$x(t) = \sqrt{2At + 1}$$

.

The same procedure can be repeated for y. We must first plug in the expression for x in the expression for v, giving

$$v = \frac{dy}{dt} = \frac{Ay}{2At + 1}.$$

Doing the same separation of variables procedure and integration will yield

$$y(t) = 3\sqrt{2At + 1}.$$

Note the similarity in $x(t)$ and $y(t)$ with the streamline. Note this similarity only exists due to steady state nature of the flow and the fact that the pathline and streamline intersected the same point.

Now to find the streakline, we revisit the integral required to find the pathline. Before we had set up an indefinite integral, but now we set a bounded integral for time between τ_0 and t. This also means we need to set a bounded integral between x_0 and x. The integrals become

$$\int_{x_0}^{x} x^{-1}\,dx = \int_{\tau_0}^{t} A\,dt$$

Solving this, plugging in $x_0 = 1$ and rearranging gives us

$$x(t) = \sqrt{2A(t-\tau_0)+1}$$

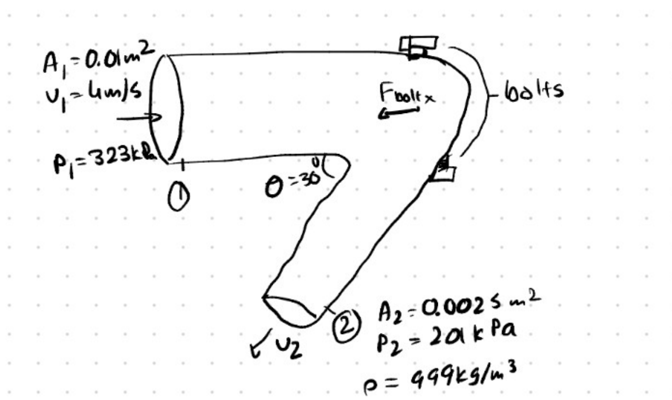
.

We repeat the same procedure for y, using the expression for v with the pathline for x plugged in. The result of this is

$$y = 3\left(\frac{2At+1}{2A\tau_0+1}\right)^{0.5}$$

To plot a streakline, you would fix a t value, and vary τ_0 from 0 to t. Conceptually this represent a series of points with different starting times, similar to a dye in an experimental setup, where each dye drop is released at a different point in time from the same initial location.

2. Water is flowing through the pipe fitting shown below. It is fixed in place by some bolts. What is the horizontal force exerted on the bolts in keeping the fitting in place?



Solution: First we will find the velocity of the water coming out the bottom of the fitting. This is a simple conservation of mass problem, with mass flow in equaling mass flow out.

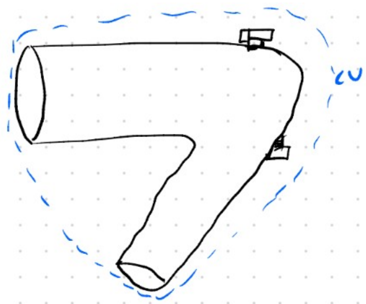
$$\rho A_1 V_1 = \rho A_2 V_2$$

Rearranging this gives $V_2 = \frac{A_1}{A_2} V_1$

Now, in order to find the force on the on the pipe, we need to use the Reynolds Transport Theorem for linear momentum in the x direction. This is given by

$$\frac{\partial}{\partial t}\int_{CV} \rho u\,dV + \int_{CS} \rho u(V\cdot\hat{n})\,dA = \Sigma F_x$$

We will now have to construct a control volume to conduct analysis with RTT. The CV is shown below. Note that the CV was creating to ensure the normal vector was in the same direction of the flow near the entrance and exit. This allows the $V \cdot \hat{n}$ and the dA portion of the equation to not take on any additional



cosine terms, simplifying the analysis.

Now let us assume steady-state, removing the time derivative portion. We can also expand out the sum of forces on the right side to include the pressure on the entrances of the pipe fitting and the force from the bolts.

$$\int_{CS} \rho u(V \cdot \hat{n})\, dA = F_x bolt + p_1 A_1 + p_2 A_2 \cos\theta$$

Note a few things. We are setting positive x as pointing to the right side. Even though the pressure is on both the entrance and the exit, they have the same sign since they both point to positive x. Lastly, since the pressure at the exit points at an angle, we only take the horizontal component of it.

Now let us expand the left hand side. We can divide our control surface into the entrance and exit of the pipe fitting, since those are the only two place where fluid is flowing. At the entrance, $V \cdot \hat{n}$ is negative since the outward pointing normal vector is in the opposite direction as the velocity. u is simply equal to V_1. This part of the equation is now $-\rho A_1 V_1^2$. At the exit, since the normal and velocity now point in the same direction, the $V \cdot \hat{n}$ is positive. However, since the flow is at an angle, the u component becomes $-V_2 \cos\theta$. Note that this is negative become u points in the negative x direction. The equation is now

$$-\rho A_1 V_1^2 - \rho A_2 V_2^2 \cos\theta = F_x bolt + p_1 A_1 + p_2 A_2 \cos\theta$$

Plugging in all the given values, gives us $F_x bolt = 3960.75N$. Since we want the force on the bolt, take the negative of this value.

Aside: If we had chosen something that resembled a square for our control volume, the control surface equation for the exit would have looked like $\rho \frac{A_1}{cos\theta}(V_2 \cos\theta)(V_2 \cos\theta)$.